# Judgment Aggregation June Project: Agenda Characterisation

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6/6/2018

(based on the slides of Ulle Endriss)

# Goals

We continue with looking into agendas that can be associated with "well-behaved" judgment aggregation.

- *Existential Agenda Characterisation*: Fix a class of aggregation rules by means of fixing some axioms. For what kinds of of agendas is there a consistent rule in that class?

See two survey papers:

C. List and C. Puppe. Judgment Aggregation: A Survey. In P. Anand, P. Pattanaik, and C. Puppe (eds.), *Handbook of Rational and Social Choice*. OUP, 2009.

U. Endriss. Judgment Aggregation. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A.D. Procaccia (eds.), *Handbook of Computational Social Choice*. CUP, 2015.

# Axioms

We use the following axioms (the last one is new!) for rules $F$:

- *Neutrality*: $N_\varphi^{\boldsymbol{J}} = N_\psi^{\boldsymbol{J}}$ implies $\varphi \in F(\boldsymbol{J}) \Leftrightarrow \psi \in F(\boldsymbol{J})$.
- *Independence*: $N_\varphi^{\boldsymbol{J}} = N_\varphi^{\boldsymbol{J}'}$ implies $\varphi \in F(\boldsymbol{J}) \Leftrightarrow \varphi \in F(\boldsymbol{J}')$.
- *Monotonicity* : $N_\varphi^{\boldsymbol{J}} \subset N_\varphi^{\boldsymbol{J}'}$ implies $\varphi \in F(\boldsymbol{J}) \Rightarrow \varphi \in F(\boldsymbol{J}')$.
- *Dictatorship*: There exists an agent $i^*$ (the dictator) such that $F(\boldsymbol{J}) = J_{i^*}$ for every profile $\boldsymbol{J}$.
  Otherwise, $F$ is *nondictatorial*.

You see how non-dictatorship is a weakening of anonymity?

# An Existential Agenda Char. Theorem

## Theorem (Nehring and Puppe, 2007)

*For $n \geqslant 3$, there exists a neutral, independent, monotonic, and nondictatorial aggregator that is complete and consistent for the agenda $\Phi$ iff $\Phi$ has the MP.*

The right-to-left direction follows from our previous Theorem: Suppose $\Phi$ has the MP. Then:

- The majority rule will be consistent and complete.
- So there exists a rule with all the required properties. ✓

Next we will prove the impossibility direction (left-to-right).

K. Nehring and C. Puppe. The Structure of Strategy-proof Social Choice. Part I: General Characterization and Possibility Results on Median Space. *Journal of Economic Theory*, 135(1):269–305, 2007.

# A very Useful Notion: Winning Coalitions

$F$ is *independent* iff there is a family of *winning coalitions* of agents $\mathcal{W}_\varphi \subseteq 2^{\mathcal{N}}$, one for each $\varphi$, s.t. $\varphi \in F(\boldsymbol{J}) \Leftrightarrow N_\varphi^{\boldsymbol{J}} \in \mathcal{W}_\varphi$.

$F$ is *independent and neutral* if furthermore we have $\mathcal{W}_\varphi = \mathcal{W}_\psi$ for all formulas $\varphi, \psi \in \Phi$. So we can simply write $\mathcal{W}$.

Now suppose $F$ is independent and neutral, and defined by $\mathcal{W}$:

- $F$ is *monotonic* iff $\mathcal{W}$ is upward closed: $C \in \mathcal{W}$ and $C \subseteq C'$ entail $C' \in \mathcal{W}$ for all $C, C' \subset \mathcal{N}$.
- $F$ is *complete* iff $C \in \mathcal{W}$ or $\overline{C} \in \mathcal{W}$ for all $C$. (why? )
- $F$ is *complement-free* iff $C \notin \mathcal{W}$ or $\overline{C} \notin \mathcal{W}$ for all $C \subseteq \mathcal{N}$.

(Note that here we assume that $\Phi$ has at least two atoms.)

# Proof Plan: Impossibility Direction

We will show that: If a rule $F$ is *neutral*, *independent*, *monotonic*, *complete*, and *consistent* for an agenda $\Phi$ violating the MP, then $F$ must be a dictatorship.

So suppose $F$ has these properties and $\Phi$ violates the MP. By *independence* and *neutrality*, there is a (single) family $\mathcal{W} \subseteq 2^{\mathcal{N}}$ of winning coalitions for $F$: $\varphi \in F(\boldsymbol{J}) \Leftrightarrow N_\varphi^{\boldsymbol{J}} \in \mathcal{W}$.

We will show that $\mathcal{W}$ is an *ultrafilter* on $\mathcal{N}$, meaning that:

(I) The *empty coalition* is not winning: $\emptyset \notin \mathcal{W}$

(II) *Closure under intersections*: $C, C' \in \mathcal{W} \Rightarrow C \cap C' \in \mathcal{W}$.

(III) *Maximality*: $C \in \mathcal{W}$ or $\overline{C} \in \mathcal{W}$.

In the end, using the finiteness of $\mathcal{N}$, we will show that $\mathcal{W} = \{C \subseteq \mathcal{N} \mid i^* \in C\}$ for some $i^* \in \mathcal{N}$, i.e., $F$ is *dictatorial*.

# (I) THE EMPTY COALITION IS NOT WINNING

We will use *monotonicity* and *complement-freeness*:

For the sake of contradiction, assume $\emptyset \in \mathcal{W}$.

- From monotonicity (i.e., closure under supersets): $\emptyset \in \mathcal{W}$ implies that $\mathcal{N} \in \mathcal{W}$.
- But now consider some profile $\boldsymbol{J}$ with $p \in J_i$ for all individuals $i \in \mathcal{N}$. (why can we take such a $\boldsymbol{J}$? ⭐)
- Then, $N_p^{\boldsymbol{J}} = \mathcal{N}$ and $N_{\neg p}^{\boldsymbol{J}} = \emptyset$.
- That is, $p \in F(\boldsymbol{J})$ and $\neg p \in F(\boldsymbol{J})$, as both $\mathcal{N}$ and $\emptyset$ are winning coalitions.
- *Contradiction* with complement-freeness. ✓

# (III) Maximality (easy first)

We will use *completeness*:

- Take any coalition $C$ and formula $\varphi$.
- Construct a profile $\boldsymbol{J}$ with $N_{\varphi}^{\boldsymbol{J}} = C$.
- From completeness: $\varphi \in F(\boldsymbol{J})$ or $\sim\varphi \in F(\boldsymbol{J})$.
- Then from $\mathcal{W}$-determination of $F$: $N_{\varphi}^{\boldsymbol{J}} \in \mathcal{W}$ or $N_{\sim\varphi}^{\boldsymbol{J}} \in \mathcal{W}$.
- From completeness and complement-freeness of $F$: $N_{\sim\varphi}^{\boldsymbol{J}} = \overline{N_{\varphi}^{\boldsymbol{J}}}$.
- Finally, all this means that $C \in \mathcal{W}$ or $\overline{C} \in \mathcal{W}$. ✓

# (II) Closure under Intersections

We use: *MP-violation*, *monotonicity*, *consistency*, *completeness*.
MP-violation: there is a *mi subset* $X = \{\varphi_1, \ldots, \varphi_k\}$ with $k \geqslant 3$.
We can construct a complete and consistent profile J with these properties (everyone accepts $k-1$ of the propositions in $X$):

- $N^{\boldsymbol{J}}_{\varphi_1} = C$.
- $N^{\boldsymbol{J}}_{\varphi_2} = C' \cup (\mathcal{N} \setminus C)$.
- $N^{\boldsymbol{J}}_{\varphi_3} = \mathcal{N} \setminus (C \cap C')$, thus $N^{\boldsymbol{J}}_{\sim\varphi_3} = C \cap C'$.
- $N^{\boldsymbol{J}}_{\psi} = \mathcal{N}$ for all $\psi \in X \setminus \{\varphi_1, \varphi_2, \varphi_3\}$.

Then, suppose that $C, C' \in \mathcal{W}$.

- $C \in \mathcal{W} \Rightarrow \varphi_1 \in F(\boldsymbol{J})$.
- (monotonicity) $C' \in \mathcal{W} \Rightarrow C' \cup (\mathcal{N} \setminus C) \in \mathcal{W} \Rightarrow \varphi_2 \in F(\boldsymbol{J})$
- (maximality) $\emptyset \notin \mathcal{W} \Rightarrow \mathcal{N} \in \mathcal{W} \Rightarrow X \setminus \{\varphi_1, \varphi_2, \varphi_3\} \subseteq F(\boldsymbol{J})$
- (consistency) $\varphi_3 \notin F(\boldsymbol{J}) \Rightarrow \sim\varphi_3 \in F(\boldsymbol{J}) \Rightarrow C \cap C' \in \mathcal{W}$. ✓

# Dictatorship

We have shown that:

(I) The *empty coalition* is not winning: $\emptyset \notin \mathcal{W}$

(II) *Closure under intersections*: $C, C' \in \mathcal{W} \Rightarrow C \cap C' \in \mathcal{W}$.

(III) *Maximality*: $C \in \mathcal{W}$ or $\overline{C} \in \mathcal{W}$.

From (I) and completeness, we have that $\mathcal{N} \in \mathcal{W}$.

*Contraction Lemma*: if $C \in \mathcal{W}$ and $|C| > 2$, then $C' \in \mathcal{W}$ for some $C' \subset C$. Proof: Let $C = C_1 \amalg C_2$. If $C_1 \notin \mathcal{W}$, then $\overline{C_1} \in \mathcal{W}$ by maximality. But then, $C \cap C_1 = C_2 \in \mathcal{W}$ by closure under intersections. ✓

Recall that $\mathcal{N}$ is *finite*. By induction (⭐): $\{i^*\} \in \mathcal{W}$ for one $i^* \in \mathcal{N}$, i.e., $\mathcal{W} = \{C \subseteq \mathcal{N} \mid i^* \in C\}$. That is, $i^*$ is a *dictator*. ✓

We just used that every ultrafilter on a finite set is *principal* !

# Relevance to Preference Aggregation

A similar characterisation result by Dokow and Holzman is particularly interesting since it can be considered a generalisation of the most famous theorem in social choice theory: *Arrow's Theorem for preference aggregation*.

To see the relevant result in judgment aggregation and for a comparison with Arrow's Theorem, consult the papers below:

E. Dokow and R. Holzman. Aggregation of Binary Evaluations. *Journal of Economic Theory*, 145(2):495–511, 2010.

F. Dietrich and C. List. Arrow's Theorem in Judgment Aggregation. *Social Choice and Welfare*, 29(1):19–33, 2007.

U. Endriss. Logic and Social Choice Theory. In A. Gupta and J. van Benthem (eds.), *Logic and Philosophy Today*. College Publications, 2011.

# Summary of Part B

Existential agenda char. theorems are of the following form:

*There exists* a nondictatorial complete and consistent *rule* meeting certain axioms $\Leftrightarrow$ the agenda has a certain *property*.

Both directions are of interest:
($\Leftarrow$) *Possibility direction*: If the agenda property holds, then "reasonable" and consistent aggregation is possible.
($\Rightarrow$) *Impossibility direction*: For structurally rich domains, all seemingly "reasonable" rules are in fact dictatorial.

Possibility is proved by providing a *concrete rule* doing the job.
Impossibility is (sometimes) proved using *ultrafilters*.

# Summary of Today

We investigated two big questions, connecting the *structure of an agenda* with the (im)possibility of *consistent* aggregation.

- First, we focused specifically on the *majority rule*.
- Then, we examined the axioms characterising the majority rule, minus anonymity, and we saw a *universal characterisation result*, also called a *safety result*.
- In the second part, we took again the majority axioms, weakening anonymity to non-dictatorship, and we saw an *existential characterisation result*.

Tomorrow, fun (and lighter) stuff is coming! ⭐ ⭐ ⭐