June Project on Judgment Aggregation
Homework 2
SOLUTION!

June 5, 2018

Today we saw two domain restrictions for judgment aggregation that ensure that the majority rule produces a consistent outcome. The purpose of this question is to understand how these two restrictions relate to each other: Show that every profile that is unidimensionally aligned is also a profile that is value-restricted.

SOLUTION

Lemma: Let $S \neq \emptyset$ be a set of subsets $I \subseteq N$ that are each intervals relative to some fixed linear order on N. Note that for all $I \in S$, $I = \{0, \ldots, k\}$ for some k or $I = \{k, \ldots, n\}$ for some k. If all elements of S are pairwise non-disjoint ($I \cap J \neq \emptyset$ for $I, J \in S$), then they are all non-disjoint ($\cap_{I \in S} I \neq \emptyset$).

Proof by induction: If $|S| = 2$, then clearly the claim holds.
Suppose now the claim holds for sets of size $m \geq 2$, and consider $S = S' \cup J$ (where $|S'| = m$). by the induction hypothesis, there is at least one agent who is in all sets in S', call this agent i. Note that J must be of the form $J = \{0, \ldots, k\}$ for some k or $J = \{k, \ldots, n\}$ for some k.
The only way that $J \cap \{i\} = \emptyset$ is if

- $i > k$ and $J = \{0, \ldots, k\}$
- $i < k$ and $J = \{k, \ldots, n\}$

In first case: $\exists I \in S'$ s.t. $i \in I$ and $J \cap \{i\} = \emptyset$ which implies $I \cap J = \emptyset$ since all $j \in I$ must be greater than k. (pairwise disjoint, so contradiction) In second case: $\exists I \in S'$ s.t. $i \in I$ and $J \cap \{i\} = \emptyset$ which implies $I \cap J = \emptyset$ since all $j \in I$ must be smaller than k. (pairwise disjoint, so contradiction)

Main proof: Suppose for contradiction that (complete and consistent profile) $\boldsymbol{J}$ is unidimentionally aligned but not value-restricted. Let X be a mi-set where value-restriction is violated. Let S be the set of intervals which correspond to the agents accepting each proposition in Y ($S = \{\{i \in N \mid p \in J_i\} \mid p \in Y\}$). Note that for all $I, J \in S$, $I \cap J \neq \emptyset$, meaning they are pairwise non-disjoint (otherwise value-restriction would not be violated for Y). But then by our lemma, we know that S has a non-empty intersection, which means there is some agent which accepts all formulas in Y (inconsistent j-set, so contradiction).