Preference Extensions in Social Choice!

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Sirin Botan

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Institute for Logic, Language and Computation
University of Amsterdam

Admin and Plan for Today

- The presentation scheduling!
- Strategyproofness for resolute vs. irresolute rules
- A number of impossibility results for various preference extensions.

Final Papers: Slowly start thinking about topics and setting up an initial meeting!

Manipulation of Voting Rules

Let's revisit the example from yesterday.

agent 1	$a \succ b \succ c$
agent 2	$b \succ c \succ a$
agent 3	$c \succ a \succ b$

Using Borda, we get $\Rightarrow \{a, b, c\}$ winning.

Recall that $\succeq_i$ is the (truthful) preference order of voter i. To distinguish between this order and the one actually submitted by the voter, we will call the submitted order her ballot.
So a ballot can be truthful (if it is the same as the true preference order of the voter) or untruthful.

Strategyproofness

We say a voter is being truthful if her ballot is her actual preference order. A voting rule is not strategyproof if an agent can submit an untruthful ballot and bring about a more preferred outcome.

- for resolute rules, it is clear what more preferred means
- for irresolute rules, we need to specify which preference extension(s) the result holds for

A voting rule f is strategyproof if there is no voter i such that there exists a profile $\mathbf{P}$ and an i-variant $\mathbf{P'}$ such that $f(\mathbf{P'}) \succ_i f(\mathbf{P})$.

We want strategyproofness because...

- voters shouldn't have to reason about strategies when voting
- if one or more voters strategise and submit untruthful ballots, the result of the election may not reflect or represent the true opinion of the electorate

Framework, Manipulation, and Strategyproofness

- set $N = \{1, \ldots, n\}$ of agents
- set A of alternatives
- $\succeq_i$ preference ranking of agent i
- A preference profile $\mathbf{P} = (\succeq_1, \ldots, \succeq_n)$
- $\mathbf{P}$ and $\mathbf{P'}$ are i-variants if the only difference is to voter i's ballot.
- $\mathcal{L}(A)^n$ set of all possible profiles

A voting rule f is a function from profiles to subsets of A.

$$f : \mathcal{L}(A)^n \to 2^A \setminus \emptyset$$

Strategyproofness for Resolute Voting Rules

Surjectivity: for any $a \in A$ there is some profile $\mathbf{P}$ such that $f(\mathbf{P}) = a$

Non-Dictatorship: no i such that $\text{top}(i) = f(\mathbf{P})$ for all $\mathbf{P} \in \mathcal{L}(A)^n$.

Theorem (Gibbard-Satterthwaite)

Any resolute voting rule for $\geqslant 3$ alternatives that is surjective and strategyproof must be a dictatorship.

Theorem (Gibbard-Satterthwaite)

There is no resolute voting rule for $\geqslant 3$ alternatives that is surjective, strategyproof and non-dictatorial.

A. Gibbard. *Manipulation of Voting Schemes: A General Result.* Econometrica 41, 1973.
M.A. Satterthwaite. *Strategy-proofness and Arrow's Conditions.* Journal of Economic Theory 10, 1975.

The Problem With Resolute Rules

agent 1	$a \succ b \succ c$
agent 2	$b \succ c \succ a$
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Q: What would a good resolute rule do here?

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Q: What would a good resolute rule do here?

Anonymity: treat all agents symmetrically.

Neutrality: treat all alternatives symmetrically.

- No resolute rule can be both anonymous and neutral.

Axioms for Irresolute rules

Recall that a resolute voting rule f is surjective if for any $a \in A$ there is some profile $\mathbf{P}$ such that $f(\mathbf{P}) = a$.

Q: What could be the equivalent axiom for irresolute rules?

Axioms for Irresolute rules

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Q: What could be the equivalent axiom for irresolute rules?

Nonimposition: for any $a \in A$ there is some profile $\mathbf{P}$ such that $a \in f(\mathbf{P})$.

Axioms for Irresolute rules

Recall that a resolute voting rule f is nondictatorial if there is no i such that $\text{top}(i) = f(\mathbf{P})$ for all $\mathbf{P} \in \mathcal{L}(A)^n$.

Q: What is a dictator in the context of irresolute rules?

Axioms for Irresolute rules

Recall that a resolute voting rule f is nondictatorial if there is no i such that $\text{top}(i) = f(\mathbf{P})$ for all $\mathbf{P} \in \mathcal{L}(A)^n$.

Q: What is a dictator in the context of irresolute rules?

A voter is a (strong) dictator if the outcome is always the singleton set with their top alternative.

A voter i is a weak dictator if their top alternative is always included in the outcome—if for all $\mathbf{P} \in \mathcal{L}(A)^n$ we have that $\text{top}(i) \in f(\mathbf{P})$.

An Early Generalisation by Kelly

Nonimposition: for any $a \in A$ there is some profile $\mathbf{P}$ such that $a \in f(\mathbf{P})$.

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Recall:

$$X \succeq^K Y \Leftrightarrow x \succeq y \text{ for all } x \in X \text{ and } y \in Y.$$

A voting rule f is Kelly-manipulable by agent i in profile $\mathbf{P}$ if there exists some $\mathbf{P'} = (\mathbf{P}_{-i}, \succeq'_i)$ where $f(\mathbf{P'}) \succ_i^K f(\mathbf{P})$.

The rule f is Kelly-strategyproof if it is not Kelly-manipulable by any agent i in any profile $\mathbf{P}$.

Theorem (Kelly, 1977)

Any voting rule that is **nonimposed***, satisfies revealed quasi-transitivity and does not have a* **weak dictator** *is Kelly-manipulable.*

J. S. Kelly. *Strategy-proofness and social choice functions without single-valuedness*. Econometrica 45, 1977.

Other Attempts at Generalising the Theorem

Condorcet Consistency: if there is an alternative that beats all others in a pairwise majority contest, it should be the sole winner.

Theorem (Gärdenfors, 1976)

Any voting rule that is anonymous, neutral and Condorcet Consistent is Gärdenfors-manipulable.

Unanimity: $f(\mathbf{P}) = \{a\}$ if a is ranked first by all voters.

Positive responsiveness: if f returns a tie and one of the tied alternatives is moved up in some voter's preference order, then this alternative becomes the sole winner.

Theorem (Barberà, 1977)

Any voting rule that satisfies unanimity and positive responsiveness, and is nondictatorial is Barberà-manipulable.

P. Gärdenfors. *Manipulation of Social Choice Functions*. Journal of Economic Theory 13, 1976.

S. Barberà. *The Manipulation of Social Choice Mechanisms That Do Not Leave Too Much to Chance*. Econometrica 45, 1977.

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Q: how is Condorcet Consistency related to nonimposition? how is unanimity related to nonimposition?

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Optimistic and Pessimistic Agents

We say a voter i is an optimist if $X \mathrel{\mathring{\succeq}_i} Y$ when there is some $a \in X$ such that $a \succeq b$ for all $b \in Y$ (compare top alternatives).

We say a voter i is a pessimist if $X \mathrel{\mathring{\succeq}_i} Y$ when for all $a \in X$ there exists some $b \in Y$ such that $a \succeq b$ (compare bottom alternatives).

A voting rule is strategyproof for optimistic (pessimistic) voters if no optimistic (pessimistic) voter can benefit from manipulating.

Theorem (Duggan and Schwartz, 2000)

Any voting rule for $\geqslant 3$ alternatives that is nonimposed and strategyproof for both optimistic and pessimistic agents must be weakly dictatorial.

Note: if the rule is resolute, this reduces to Gibbard-Satterthwaite.

J. Duggan and T. Schwartz. *Strategic Manipulation without Resoluteness or Shared Beliefs: Gibbard-Satterthwaite Generalized*. Social Choice and Welfare, 2000.

A.D. Taylor. *The Manipulability of Voting Systems*. The American Mathematical Monthly, 2002.

Duggan-Schwartz Theorem

Most prominent result of its kind. But the strategyproofness notion is quite demanding.

Theorem (Duggan and Schwartz, 2000)

Any voting rule for $\geqslant 3$ alternatives that is nonimposed and strategyproof for both optimistic and pessimistic agents must be weakly dictatorial.

D-S manipulability: there is some agent i, and profile $\mathbf{P}$ with i-variant $\mathbf{P'}$ such that: for any lottery over $f(\mathbf{P'})$, and any lottery over $f(\mathbf{P})$ there is some utility function u_i such that the expected utility of the $f(\mathbf{P'})$-lottery exceeds the expected utility of the $f(\mathbf{P})$-lottery.

Theorem (Duggan and Schwartz, 2000)

No voting rule satisifies DS strategyproofness, surjectivity, non-dictatorship, and residual resoluteness.[1]

J. Duggan and T. Schwartz. *Strategic Manipulation without Resoluteness or Shared Beliefs: Gibbard-Satterthwaite Generalized*. Social Choice and Welfare, 2000.

[1] Requires singletons to be returned in certain scenarios.

A Comparison

- **Nonimposition** + **no weak dictator** + revealed quasi-transitivity $\Rightarrow$ Kelly-manipulable.
- **Condorcet Consistency** + anonymity + neutrality $\Rightarrow$ Gärdenfors-manipulable.
- **Unanimity** + **nondictatorial** + positive responsiveness $\Rightarrow$ Barberà-manipulable.
- **Nonimposition** + **no weak dictator** $\Rightarrow$ manipulable by optimistic or pessimistic agents.

Weaker extensions give stronger impossibility results in general. , but it is difficult to compare these results based on strength of the extension, as they need different extra assumptions to go through.

Summary and Info for Next Week

We saw the Gibbard-Satterthwaite theorem and various attempts at generalising it to irresolute rules. We'll see results relating to other extensions in some of the presentations next week—including more hopeful ones!

Presentations:

- Go slow, explain everything, don't assume the audience knows.
- More like a lecture and less like a seminar talk.

Final Papers:

- groups of 2-3 unless you desperately want to work on your own
- set up a first meeting with me to discuss ideas (I'm going to make a google sheet for this)