Preference Extensions in Social Choice

January 2021 MoL Project

Sirin Botan

05/01/21

Institute for Logic, Language and Computation
University of Amsterdam

Plan for Today

- the response in the literature to the Kannai-Peleg impossibility
  - similar impossibilities
  - extensions that are not connex
  - weakening axioms to circumvent result
- A first look at voting
- paper presentations info

Yesterday, we saw the Kannai-Peleg Theorem.

(DOM) $a \succ b$ for all $b \in X \Rightarrow \{a\} \cup X \mathrel{\mathring{\succ}} X$

(DOM) $b \succ a$ for all $b \in X \Rightarrow X \mathrel{\mathring{\succ}} X \cup \{a\}$

(IND) $X \mathrel{\mathring{\succ}} Y$ implies $X \cup \{a\} \mathrel{\mathring{\succeq}} Y \cup \{a\}$ for all $a \in A \setminus (X \cup Y)$

Theorem (Kannai and Peleg, 1984)

There exists no extension satisfying both dominance and independence.

One way around the Impossibility

We define the minmax dominance extension. Note that this is not a connex relation.

$$X \mathring{\succeq} Y \Leftrightarrow [\max(X) \succeq \max(Y) \text{ and } \min(X) \succeq \min(Y)]$$

Q: why is it not connex? ($\{a, c\} \mathring{\succeq} \{b\}$? $\{b\} \mathring{\succeq} \{a, c\}$?)

Q: Is independence satisfied? Is dominance satisfied?

- (IND) if $X \mathring{\succ} Y$, then either $\max(X) \succ \max(Y)$ or $\min(X) \succ \min(Y)$, so "worst case" we get $X \cup \{a\} \mathring{\succeq} Y \cup \{a\}$.
- (DOM) if $a \succ b$ for all $b \in X$, then $\min(X \cup \{a\}) = \min(X)$ and $\max(X \cup \{a\}) \succ \max(X)$.
- (DOM) if $b \succ a$ for all $b \in X$, then $\max(X \cup \{a\}) = \max(X)$, and $\min(X) \succ \min(X \cup \{a\})$.

Another Impossibility

Simple Dominance applies to expansions of singleton sets by one element:

$$a \succ b \Rightarrow [\{a\} \mathrel{\mathring{\succ}} \{a, b\} \text{ and } \{a, b\} \mathrel{\mathring{\succ}} \{b\}.]$$

Strict Independence: for $a \in A \setminus (X \cup Y)$,

$$X \mathrel{\mathring{\succ}} Y \Rightarrow X \cup \{a\} \mathrel{\mathring{\succ}} Y \cup \{a\}.$$

No assumption that $\mathring{\succeq}$ is connex or transitive!

Theorem (Barberà and Pattanaik, 1984)

There exists no extension satisfying simple dominance and strict independence.[1]

S. Barberà and P.K. Pattanaik. *Extending an Order on a Set to the Power Set: Some Remarks of Kannai and Peleg's Approach*. Journal of Economic Theory 32, 1984.

[1] $|A| \geqslant 3$ and $\succeq$ a linear order.

Another Impossibility

Simple Dominance applies to expansions of singleton sets by one element:

$$a \succ b \Rightarrow [\{a\} \mathrel{\mathring{\succsim}} \{a, b\} \text{ and } \{a, b\} \mathrel{\mathring{\succsim}} \{b\}.]$$

Strict Independence: for $a \in A \setminus (X \cup Y)$,

$$X \mathrel{\mathring{\succsim}} Y \Rightarrow X \cup \{a\} \mathrel{\mathring{\succsim}} Y \cup \{a\}.$$

Proof.

For $\bot$ suppose S-DOM and S-IND. We have $a \succ b \succ c$.

- (S-DOM) $\{a\} \mathrel{\mathring{\succsim}} \{a, b\}$
- (S-IND) $\{a, c\} \mathrel{\mathring{\succsim}} \{a, b, c\}$
- (S-DOM) $\{b, c\} \mathrel{\mathring{\succsim}} \{c\}$
- (S-IND) $\{a, b, c\} \mathrel{\mathring{\succsim}} \{a, c\}$

We have a contradiction. □

S. Barberà and P.K. Pattanaik. *Extending an Order on a Set to the Power Set: Some Remarks of Kannai and Peleg's Approach*. Journal of Economic Theory 32, 1984.

Maxmin-based Extensions

Simple Dominance applies to expansions of singleton sets by one element. Restricted Independence restricts attention to comparisons of two-element sets.

An extension is maxmin-based iff there is an ordering $\mathring{\succeq}_{1,2}$ on $\mathcal{A}_{1,2}$ satisfying simple dominance and restricted independence s.t.

$$X \mathrel{\mathring{\succeq}} Y \Leftrightarrow \{\max(X), \min(X)\} \succeq_{1,2} \{\max(Y), \min(Y)\}$$

Theorem (Barberà, Barrett, and Pattanaik, 1984)

$\mathring{\succeq}$ satifies simple dominance and independence iff it is maxmin-based.

Recover the order from restriction to singletons and two-element sets. This result shows that if we weaken dominance, we can "circumvent" the impossibility result. Also characterises class of maxmin-based extensions!

S. Barberà, C.R. Barrett, and P.K. Pattanaik. *On Some Axioms for Ranking Sets of Alternatives*. Journal of Economic Theory 33, 1984.

Two Maxmin-based Extensions: Minimax and Maximax

As always $a \succ b \succ c$.

- $X \succeq_{minimax} Y \Leftrightarrow \min(X) \succ \min(Y)$
  $\quad$ or $[\min(X) = \min(Y), \max(X) \succeq \max(Y)]$
  - $\{a, c\} \overset{\circ}{\succeq} \{b, c\}$?
  - $\{b\} \overset{\circ}{\succ} \{a, c\}$?
- $X \succeq_{maximax} Y \Leftrightarrow \max(X) \succ \max(Y)$
  $\quad$ or $[\max(X) = \max(Y), \min(X) \succeq \min(Y)]$
  - $\{a, c\} \overset{\circ}{\succeq} \{b, c\}$?
  - $\{b\} \overset{\circ}{\succeq} \{a, c\}$?

Can be interpreted as attitude towards uncertainty. Minimax is uncertainty aversion, and maximax is uncertainty appeal or more risk-taking.

Note: indifferent if $\max(X) = \max(Y)$ and $\min(X) = \min(Y)$.

Minimax and Maximax Characterisations*

NOTE: These Thms. are incorrect as these extensions do not satisfy IND.

Top Monotonicity: $\Rightarrow \{a, c\} \mathrel{\mathring{\succ}} \{b, c\}$.

Uncertainty Aversion: $\Rightarrow \{b\} \mathrel{\mathring{\succ}} \{a, c\}$.

Theorem (Bossert, Pattanaik, and Wu, 1994)

$\mathring{\succeq}$ satisfies simple dominance, independence, uncertainty aversion, and top monotonicity iff $\mathring{\succeq} = \succeq_{minimax}$

Bottom Monotonicity: $a \succ b \succ c \Rightarrow \{a, b\} \mathrel{\mathring{\succ}} \{a, c\}$.

Uncertainty Appeal: $\Rightarrow \{a, b\} \mathrel{\mathring{\succ}} \{b\}$.

Theorem (Bossert, Pattanaik, and Wu, 1994)

$\mathring{\succeq}$ satisfies simple dominance, independence, uncertainty appeal, and bottom monotonicity iff $\mathring{\succeq} = \succeq_{maximax}$

W. Bossert, P.K. Pattanaik, and Y. Xu. *Choice Under Complete Uncertainty: Axiomatic Characterizations of some Decision Rules*. Journal of Economic Theory 63, 1994.

Minimax and Maximin Characterisations*

$a \succ b \succ c$...

Theorem (Bossert, Pattanaik, and Wu, 2000)

$\mathring{\succeq}$ satisfies simple dominance, independence, uncertainty aversion, and top monotonicity iff $\mathring{\succeq} = \succeq_{minimax}$

Theorem (Bossert, Pattanaik, and Wu, 2000)

$\mathring{\succeq}$ satisfies simple dominance, independence, uncertainty appeal, and bottom monotonicity iff $\mathring{\succeq} = \succeq_{maximax}$

Q: why is IND not satisfied?

W. Bossert, P.K. Pattanaik, and Y. Xu. *Choice Under Complete Uncertainty: Axiomatic Characterizations of some Decision Rules*. Journal of Economic Theory 63, 1994.
R. Arlegi. A note on Bossert, Pattanaik and Xus Choice under complete uncertainty: axiomatic characterization of some decision rules. Economic Theory 22. 2003.

Minimax and Maximin Characterisations*

$a \succ b \succ c \ldots$

Theorem (Bossert, Pattanaik, and Wu, 2000)

$\mathring{\succeq}$ satisfies simple dominance, independence, uncertainty aversion, and top monotonicity iff $\mathring{\succeq} = \succeq_{minimax}$

Theorem (Bossert, Pattanaik, and Wu, 2000)

$\mathring{\succeq}$ satisfies simple dominance, independence, uncertainty appeal, and bottom monotonicity iff $\mathring{\succeq} = \succeq_{maximax}$

Q: why is IND not satisfied?

$\{2, 5\} \succ_{max} \{3, 4\}$

IND $\Rightarrow \{1, 2, 5\} \succeq_{max} \{1, 3, 4\}$ 😬

W. Bossert, P.K. Pattanaik, and Y. Xu. *Choice Under Complete Uncertainty: Axiomatic Characterizations of some Decision Rules*. Journal of Economic Theory 63, 1994.

R. Arlegi. A note on Bossert, Pattanaik and Xus Choice under complete uncertainty: axiomatic characterization of some decision rules. Economic Theory 22. 2003.

Extensions Satisfying Dominance: Leximin and Leximax

An extension satisfies dominance (DOM) if for all $X \in \mathcal{A}$, for all $a \in A$

1. $a \succ b$ for all $b \in X \Rightarrow \{a\} \cup X \mathrel{\mathring{\succ}} X$
2. $b \succ a$ for all $b \in X \Rightarrow X \mathrel{\mathring{\succ}} X \cup \{a\}$

Leximin first looks at the worst elements of X and Y.

- If $\min(X) \succ \min(Y)$ then $X \mathrel{\mathring{\succ}} Y$,
- else, eliminate $\min(X)$ and $\min(Y)$ and continue the procedure.
  1. $\{a, c, d\}$ vs. $\{b, c, d\}$
  2. $\{a, c\}$ vs. $\{b, c\}$
  3. $\mathbf{\{a\}}$ vs. $\{b\}$

Leximax does the same with $\max(X)$ and $\max(Y)$.

More emphasis on min or max elements compared with maximin and maximax: leximin never looks at max.

Extensions Satisfying Dominance: Leximin and Leximax

Theorem (Pattanaik and Peleg, 1984)

$\mathring{\succeq}$ satisfies dominance, neutrality, bottom independence, and disjoint independence iff $\mathring{\succeq} = \succeq^L_{min}$

Theorem (Pattanaik and Peleg, 1984)

$\mathring{\succeq}$ satisfies dominance, neutrality, top independence, and disjoint independence iff $\mathring{\succeq} = \succeq^L_{max}$

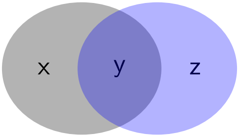
P.K. Pattanaik, and B. Peleg. *An Axiomatic Characterization of the Lexicographic Maximin Extension of an Ordering Over a Set to the Power Set.* Social Choice and Welfare 1, 1984.

Fishburn Extension

Let's look at extensions defined for use in voting.

$X \succeq^F Y \Leftrightarrow$

1. $x \succeq y$ for all $x \in X \setminus Y$ and $y \in Y \cap X$, and
2. $y \succeq z$ for all $y \in X \cap Y$ and $z \in Y \setminus X$, and
3. $x \succeq z$ for all $x \in X \setminus Y$ and $z \in Y \setminus X$.



Suppose $a \succ b \succ c \succ d$

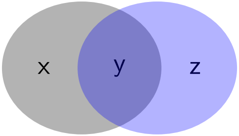
- $\{a, b, c\}$ or $\{b, c, d\}$?
- $\{a, b\}$ or $\{a, c\}$?

P.C. Fishburn. *Even-chance Lotteries in Social Choice Theory*. Theory and Decision 3, 1972.

Fishburn Extension

Let's look at extensions defined for use in voting.

$X \succeq^F Y \Leftrightarrow$
1. $x \succeq y$ for all $x \in X \setminus Y$ and $y \in Y \cap X$, and
2. $y \succeq z$ for all $y \in X \cap Y$ and $z \in Y \setminus X$, and
3. $x \succeq z$ for all $x \in X \setminus Y$ and $z \in Y \setminus X$.



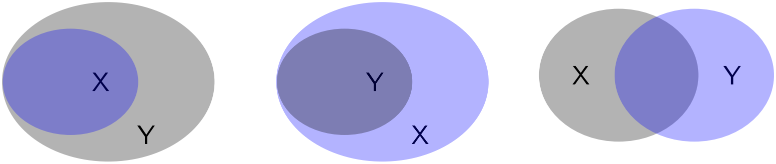
Interpretation: tie-breaker with linear, but unknown preferences.
$\{a, b\} \mathring{\not\succeq}^F \{a, c\}$ because ties may be broken in the order b, a, c.

P.C. Fishburn. *Even-chance Lotteries in Social Choice Theory*. Theory and Decision 3, 1972.

Gärdenfors Extension

$X \succeq^G Y \Leftrightarrow$

1. $X \subset Y$ and $x \succeq y$ for all $x \in X$ and $y \in Y \setminus X$ OR
2. $Y \subset X$ and $x \succeq y$ for all $x \in X \setminus Y$ and $y \in Y$ OR
3. $X \not\subset Y$, $Y \not\subset X$, and $x \succeq y$ for all $x \in X \setminus Y$ and $y \in Y \setminus X$



(*i*) (*ii*) (*iii*)

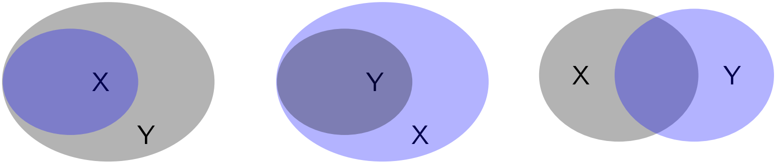
- $\{a, b\}$ or $\{a, c\}$?
- $\{a, b, c\}$ or $\{a, c\}$?

P. Gärdenfors. *Manipulation of Social Choice Functions*. Journal of Economic Theory 13, 1976.

Gärdenfors Extension

$X \succeq^G Y \Leftrightarrow$

1. $X \subset Y$ and $x \succeq y$ for all $x \in X$ and $y \in Y \setminus X$ OR
2. $Y \subset X$ and $x \succeq y$ for all $x \in X \setminus Y$ and $y \in Y$ OR
3. $X \not\subset Y$, $Y \not\subset X$, and $x \succeq y$ for all $x \in X \setminus Y$ and $y \in Y \setminus X$



(i) (ii) (iii)

Note that this extension satisfies DOM. You will sometimes see DOM referred to as the Gärdenfors principle.

P. Gärdenfors. *Manipulation of Social Choice Functions*. Journal of Economic Theory 13, 1976.

What is Voting?

agent 1	$a \succ b \succ c$
agent 2	$b \succ a \succ c$
agent 3	$b \succ c \succ a$

Example of a rule: Borda. Gives 2 points to alternative each time it is ranked first and 1 point if it is ranked second.

$a : 3, b : 5, c : 2$, so $\{a\}$ is the winning set.

Framework

- set $N = \{1, \ldots, n\}$ of agents
- set A of alternatives
- $\succeq_i$ preference ranking of agent i
- A preference profile $\mathbf{P} = (\succeq_1, \ldots, \succeq_n)$
- $\mathcal{L}(A)^n$ set of all possible profiles

An irresolute voting rule f is a function from profiles to subsets of A.

$$f : \mathcal{L}(A)^n \rightarrow 2^A \setminus \emptyset$$

Manipulation of Voting Rules

agent 1	$a \succ c \succ b$
agent 2	$b \succ a \succ c$
agent 3	$b \succ c \succ a$
agent 4	$c \succ b \succ a$

Suppose we use the plurality rule, which selects as winners those alternatives that appear most at the top $\Rightarrow \{b\}$ is winning set.

Q: What happens if agent 1 flips a and c?

Q: We have that $X \succeq^K Y \Rightarrow X \succeq^F Y \Rightarrow X \succeq^G Y$. If a rule is Kelly-manipulable, what does this imply for Fishburn and Gärdenfors?

We will look at this problem in depth on Thursday!

Manipulation of Voting Rules

agent 1	$a \succ b \succ c$
agent 2	$b \succ c \succ a$
agent 3	$c \succ a \succ b$

Let's use Borda again $\Rightarrow \{a, b, c\}$ winning.

Q: What happens when agent 1 flips a and b?

Q: We have that $X \succeq^K Y \Rightarrow X \succeq^F Y \Rightarrow X \succeq^G Y$. If a rule is Kelly-manipulable, what does this imply for Fishburn and Gärdenfors?

We will look at this problem in depth on Thursday!

Paper Presentations

- Geist and Endriss. *Automated Search for Impossibility Theorems in Social Choice Theory: Ranking Sets of Objects*. 2011.
  - SAT-solver used for Kannai-Peleg and related results
- Maly et al. *Preference Orders on Families of Sets—When Can Impossibility Results Be Avoided?* 2018.
  - looks at impossibility result when limiting attention to sets of a certain type
- Maly. *Lifting Preferences over Alternatives to Preferences over Sets of Alternatives: The Complexity of Recognizing Desirable Families of Sets.* 2020.
  - looks at the complexity of identifying certain types of sets (ex. those from Maly, 2018 *)

Paper Presentations

- Brandt. *Set-Monotonicity Implies Kelly-Strategyproofness.* 2015.
  - identifies voting rules that are Kelly-SP.
- Aziz et al. *On the Incompatibility of Efficiency and Strategyproofness in Randomized Social Choice.* 2014.
  - impossibility-style result (building on yet another one), using preference extensions. This one concerned with whether you can have a SP voting rule that is also efficient/Pareto optimal.
- Brandt et al. *On the Indecisiveness of Kelly-Strategyproof Social Choice Functions*. 2020.
  - more in detail on Kelly-SP voting rules
- Brandl et al. *Strategic Abstention Based on Preference Extensions: Positive Results and Computer-Generated Impossibilities*. 2015.
  - Looks at abstention rather than submitting untruthful ranking.

Some Notes for Presentations

- The papers vary in length, but long does not mean difficult to read.
- It's ok if you don't fully understand everything in the paper.
- Spend some time thinking about what aspects to present
  - some proofs are interesting, some you should shield us from
  - some papers have a lot of new terminology and concepts and you may want to spend a substantial time on that (ex. SAT solving papers)

We have 6 one-hour slots next week. Tue 10-11, 11-12, Thu 16-17, 17-18, and Fri 14-15, 15-16. Choose partner(s), paper, and slot then email me. I'll update the website as you pick slots.

If you have any questions (big or small), please email me!

Last Slide

- We saw a variant of the Kannai-Peleg Theorem
- we saw that we can weaken "output" requirements (non-connex preference relation over sets)
- we saw that we can weaken either dominance and independence to get around the result
- we saw the Fishburn and Gärdenfors extensions
- we had a first look at strategyproofness in voting

Thursday we dive into how extensions appear in strategyproofness results.