Preference Extensions (in Social Choice)

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Slides will be up on project website after lectures. 50 + 50 min lectures.

- 2 more lectures this week on Tu & Th
- Presentations
  - Read and present in pairs
  - You will be able to choose a paper and a slot after today's lecture
- Final papers: too early to talk about!

Today, we'll look at the formal framework, some properties of set extensions, and an impossibility result: Kannai-Peleg Theorem

What is (the point of) a Preference Extension?

We want to relate—or extend—the preferences over objects to preferences over sets. Why? To reason about preferences over sets withough having to ask for them directly.

$$\begin{tikzcd}
a \arrow[d] & & \{a, b\} \arrow[d] \\
b \arrow[d] \arrow[rr, "\text{extension}"'] & & \{\vdots\} \arrow[d] \\
c & & \{c\}
\end{tikzcd}$$

prefs over items

prefs over sets

S. Barberà, W.Bossert, and P.K. Pattanaik. *Ranking Sets of Objects*. Handbook of Utility Theory, 2004.

What is the point (of a Preference Extension)?

Three main interpretations.

- Opportunity sets: agent may choose an item from each set however she wants.
  - Focus on freedom of choice & flexibility.
  - $a \succ b \succ c$, but it's very close between a and b. I may then not be indifferent between $\{a\}$ and $\{a, b\}$.
- Sets as final outcomes. Agent receives whole set.
  - applications: matching, committee elections, coalition formation...
- **Sets as mutually exclusive alternatives.**

Sets as Final Outcomes

Each set represent a bundle of items/objects that the agent receives.
Let's look at some properties in this setting.

Responsiveness: extension "responds" to swapping less preferred for more preferred. If you take out a and add b then this is preferred if and only if b is preferred to a.

Separability says adding any "desirable" item is good. An item a is desirable if $\{a\} \succ \emptyset$.

Q: Do these make sense if the sets represent mutually exclusive outcomes?

Mutually Exclusive Alternatives

Each set represents a number of alternatives, only one of which will materialise in the end. Agent (usually) does not know how the tie is broken.

- Complete uncertainty: agent knows possible outcomes, but cannot assign probabilities to those outcomes/alternatives.
- Most relevant to voting, or other collective decision-making frameworks (ex. Judgment Aggregation, Multiwinner Voting)

This setting will be our main focus, and is also the predominant interpretation of set extensions in the literature.

Quick History

- 1970s: economists define extensions for studying strategyproofness in social choice (Kelly, Barberà, Fishburn, Gardenfors, Pattanaik...)
- 1980s: set extensions studied independent of any application (Kannai, Peleg, Barberà, Pattanaik, Bossert, Dutta...)
- 1990s-2010s: various work from social choice theorists, mostly economists. Variations of existing results, characterisations.
- 2010s and on: renewed interest from COMSOC community, more computational (Brandt, Geist, Maly...)

Note: this is not a large topic in the literature. We will cover essentially all relevant results. I will do my very best to situate them in the larger context of the field.

Framework

$\succeq$ is antisymmetric if $a \succeq b$ and $b \succeq a \Rightarrow a = b$,
...transitive if $a \succeq b$ and $b \succeq c \Rightarrow a \succeq c$,
...connex if $a \succeq b$ or $b \succeq a$ for all $a, b \in A$,
...reflexive if $a \succeq a$.

- Nonempty (finite) set of alternatives A
- A preference order $\succeq \subseteq A \times A$ is a linear order[1] on A.
- $\mathcal{A} = 2^A \setminus \{\emptyset\}$ is the set of nonempty subsets of A
- A preference order over sets $\mathring{\succeq} \subseteq \mathcal{A} \times \mathcal{A}$ is a binary relation on $\mathcal{A}$.

$$a \succ b \text{ iff } a \succeq b \text{ and } b \not\succeq a$$
$$a \sim b \text{ iff } a \succeq b \text{ and } b \succeq a$$

[1] It is antisymmetric, transitive, and connex.

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A preference extension (or set extension) is a function

$$e : A \times A \mapsto \mathcal{A} \times \mathcal{A}$$

[1] It is antisymmetric, transitive, and connex.

Example: Chocolate Bars

$\succ$


$\succ$


$\succ$


Q: Which bundle of chocolates should I choose, and why?

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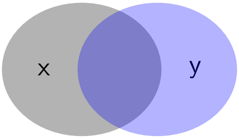
Concrete Extension: Kelly

first axiom is the so-called extension rule: (EXT) $a \succeq b \Leftrightarrow \{a\} \mathrel{\mathring{\succeq}} \{b\}$.
Let $\max(X)$ be the top element of X, and $\min(X)$ the bottom element of X (technically defined relative to a ranking $\succeq$).

(MAX) $\{\max(X)\} \mathrel{\mathring{\succeq}} X$ (MIN) $X \mathrel{\mathring{\succeq}} \{\min(X)\}$

These three axioms together define the Kelly Extension:

$$X \succeq^K Y \Leftrightarrow x \succeq y \text{ for all } x \in X \text{ and } y \in Y.$$



Q: if $a \succ b \succ c \succ d$:

- $\{a, b\}$ or $\{c, d\}$?, $\{b, c\}$ or $\{c, d\}$?, $\{a, b, c\}$ or $\{b, c, d\}$?

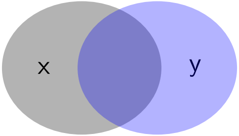
J. S. Kelly. *Strategy-proofness and social choice functions without single-valuedness*. Econometrica 45, 1977.

Concrete Extension: Kelly

(EXT)	(MAX)	(MIN)
$a \succeq b \Leftrightarrow \{a\} \mathrel{\mathring{\succeq}} \{b\}$	$\{\max(X)\} \mathrel{\mathring{\succeq}} X$	$X \mathrel{\mathring{\succeq}} \{\min(X)\}$

$$X \succeq^K Y \Leftrightarrow x \succeq y \text{ for all } x \in X \text{ and } y \in Y.$$

- $\{a, b\} \succ^K \{b\}$ (MIN), $\{b\} \succ^K \{c\}$ (EXT), $\{c\} \succ^K \{c, d\}$ (MAX) $\Rightarrow \{a, b\} \mathrel{\mathring{\succ}} \{c, d\}$



Note: if $|X \cap Y| > 1$ the sets are incomparable under $\succ^K$.

J. S. Kelly. *Strategy-proofness and social choice functions without single-valuedness*. Econometrica 45, 1977.

Two Axioms: Dominance and Independence

An extension satisfies dominance (DOM) if for all $X \in \mathcal{A}$, for all $a \in A$

1. $a \succ b$ for all $b \in X \Rightarrow \{a\} \cup X \mathrel{\mathring{\succ}} X$
2. $b \succ a$ for all $b \in X \Rightarrow X \mathrel{\mathring{\succ}} X \cup \{a\}$

An extension satisfies independence (IND) if, for all $X, Y \in \mathcal{A}$ for all $a \in A \setminus (X \cup Y)$

$$X \mathrel{\mathring{\succ}} Y \text{ implies } X \cup \{a\} \mathrel{\mathring{\succeq}} Y \cup \{a\}$$

Q: does the Kelly extension satisfy dominance? independence?

(Hint: suppose $a \succ b \succ c \succ d$, and $X = \{b, c\}$, $Y = \{c, d\}$.)

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Q: does the Kelly extension satisfy dominance? independence?

(Hint: suppose $a \succ b \succ c \succ d$, and $X = \{b, c\}$, $Y = \{c, d\}$.)
$\{a, b, c\} \succ^K \{b, c\}$? $\{a, b, c\} \succ^K \{a, c, d\}$?

Kannai-Peleg Theorem

$\mathring{\succeq}$ is a reflexive, transitive and connex relation over $\mathcal{A}$.

$X = \{1, 2, \ldots, k\}$ where $i \succ i + 1$ for all $i \in \{1, \ldots, k-1\}$.

Lemma (Kannai and Peleg, 1984)

Dominance + independence $\Rightarrow X \mathring{\sim} \{\max(X), \min(X)\}$.

One of the first papers to study set extensions as a problem of interest independent of applications. This paper is short and a lovely read.

Y. Kannai and B. Peleg. *A note on the extension of an order on a set to the power set*. Journal of Economic Theory 32, 1984.

Proof.

- If $|X| < 3$, then $X = \{\max(X), \min(X)\}$ (and $\mathring{\succeq}$ is reflexive).
- If $|X| = k \geqslant 3$
  - (DOM) $X \setminus \{k, k-1\} \mathring{\succeq} X \setminus \{k\}$
  - (DOM) $X \setminus \{k, k-1, k-2\} \mathring{\succeq} X \setminus \{k, k-1\}$
  - Repeat, and use transitivity: $\{1\} \mathring{\succeq} X \setminus \{k\}$.

  - (DOM) $\{k, k-1\} \mathring{\succeq} \{k\}$
  - (DOM) $\{k, k-1, k-2\} \mathring{\succeq} \{k, k-1\}$
  - Repeat, and use transitivity: $X \setminus \{1\} \mathring{\succeq} \{k\}$.

- (IND) $\{1, k\} \mathring{\succeq} X$ (add k)
- $X \mathring{\succeq} \{1, k\}$ (add 1)

So $\{\min(X), \max(X)\} \mathring{\succeq} X$ and $X \mathring{\succeq} \{\min(X), \max(X)\}$.

$X \mathring{\sim} \{\min(X), \max(X)\}$ $\square$

Kannai-Peleg Theorem

Theorem (Kannai and Peleg, 1984)

There exists no extension satisfying both dominance and independence.[2]

Proof.

Suppose DOM + IND (for $\perp$). Let $1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6$.

If $\{3\} \mathrel{\mathring{\succ}} \{2,5\}$:

- (Lemma) $\{3,4,5,6\} \mathrel{\mathring{\sim}} \{3,6\}$, (IND) $\{3,6\} \mathrel{\mathring{\succeq}} \{2,5,6\}$
  $\Rightarrow \{3,4,5,6\} \mathrel{\mathring{\succeq}} \{2,5,6\}$ (transitivity).
- (Lemma) $\{2,5,6\} \sim \{2,6\} \mathrel{\mathring{\sim}} \{2,3,4,5,6\}$
  $\Rightarrow \{3,4,5,6\} \mathrel{\mathring{\succeq}} \{2,3,4,5,6\}$ Contradicts DOM.

If $\{2,5\} \mathrel{\mathring{\succeq}} \{3\}$:

- (DOM) $\{3\} \mathrel{\mathring{\succ}} \{4\} \Rightarrow \{2,5\} \mathrel{\mathring{\succeq}} \{4\}$ ($\mathring{\succeq}$ is transitive)
- (IND) $\{1,2,5\} \mathrel{\mathring{\succeq}} \{1,4\}$
- (Lemma) $\{1,2,5\} \sim \{1,2,3,4,5\}$, and $\{1,4\} \mathrel{\mathring{\succ}} \{1,2,3,4\}$
- $\Rightarrow \{1,2,3,4,5\} \mathrel{\mathring{\succeq}} \{1,2,3,4\}$ Contradicts DOM.

$\square$

[2] $|A| \geqslant 6$ and $\succeq$ is a linear order on A, $\mathring{\succeq}$ is reflexive, transitive and connex.

What we did today...

Today has been an introduction to set extensions. We saw

- Possible interpretations for preference extensions
- The Kelly extension
- Two central axioms: dominance and independence
- The Kannai-Peleg Impossibility Theorem

...what's coming next.

Tomorrow we'll continue talking about set extensions.

- reactions to the impossibility
- Prominent extensions, characterisation results
- Class of extensions based on min and max elements

Thursday we'll look at how extensions have been used in social choice.