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Notes, Comments, and Letters to the Editor

A Note on the Extension of an Order on a Set to the Power Set*

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The problem of extending an order on a set to the power set arises quite frequently in social choice theory. An axiomatic treatment of this problem is provided by this note. In particular, we show that the combination of Gärdenfors' principle of extension with a very mild monotonicity requirement leads to an impossibility result. *Journal of Economic Literature* Classification Number: 025.

1. INTRODUCTION

Consider a voting situation where the outcome of the voting is determined by a social choice *correspondence*. Then, while the result of the voting is determined by the (declared) preferences of the voters over alternatives, a voter cannot compare possible outcomes unless he has an order relation over *sets* of alternatives. Thus, there is a need to "extend" the voters' preferences from the set Ω of alternatives to 2^{Ω}. As far as we know, Fishburn [1] was the first to consider explicitly preferences of voters over sets of alternatives. Fishburn's paper also contains an axiomatic characterization of preference orders over sets of alternatives, which are derived from utility maximization with respect to even-chance lotteries. Recent systematic studies of extension of preferences from Ω to 2^{Ω} are Gärdenfors [3] and Packard [4]. In particular, Packard [4] contains an axiomatic characterization of five well-known methods of extension. (The reader is referred to Gärdenfors [3] and

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Packard [4] for further details.) Also, Section 2.3 in Pattanaik [6] is devoted to a detailed investigation of the relationship between preferences over alternatives and preferences over sets of alternatives.

Our approach to the extension problem is somewhat different from that of the above-mentioned investigations. We propose an axiomatic treatment of the extension problem itself, while the previous authors provided, mainly, axiomatic characterization of known methods of extension. Specifically, we show in this note that the combination of Gärdenfors principle of extension (GP) (see Gärdenfors [2]) with a very mild monotonicity property (M), leads to an impossibility theorem (see Section 2). As has been pointed out to us by Fishburn (private communication), it is quite easy to see that (GP) and (M) exclude the "additive" and the "averaging" approaches to the extension problem (respectively). The surprising fact is that (GP) and (M) exclude *all* possible ways of extension, including, inter alia, Pattanaik's maximin rule which is neither "additive" nor is it an averaging-method. Our result seems to indicate that the nonexistence of a ("practical") "canonical" extension is due to the nonexistence of extensions which are "reasonable" and their applicability is not limited to a specific problem. However, we emphasize that many specific problems of extension of preferences do have satisfactory solutions. A remarkable example is Packard's plausibility orderings (see Packard [5]).

2. An Impossibility Theorem

Let Ω be a set and let R be a linear ordering of Ω (i.e., R is a complete, transitive and antisymmetric binary relation on Ω). (Intuitively, Ω is a set of alternatives and R is the order relation of a decision-maker on Ω.) We denote by 2^Ω the set of all *non*-empty subsets of Ω. Let further $\succsim$ be a reflexive binary relation on 2^Ω. (Intuitively, $\succsim$ is an "extension" of R to 2^Ω in the following sense. A selection, or a tie-breaking rule, is a (possibly stochastic) device ρ which selects one element $\rho(B) \in B$ for every $B \in 2^\Omega$. Let $A, B \in 2^\Omega$. $A \succsim B$ if for every "reasonable" selection ρ the decision-maker prefers ρ to choose from A rather than from B.) For $A, B \in 2^\Omega$ we denote $A \succ B$ if $A \succsim B$ and not $B \succsim A$, and $A \sim B$ if $A \succsim B$ and $B \succsim A$. We shall be interested in the following relation between R and $\succsim$. Let A be a finite subset of Ω. We denote by $\max(A)(\min(A))$ the greatest (smallest) member of A in the order R. We are now able to formulate *Gärdenfors principle* (GP) (see Gärdenfors [2]).

(GP) Let A be a non-empty finite subset of Ω and let $x \in \Omega - A$. If $xR \max(A)$ then $A \cup \{x\} \succ A$, and if $\min(A)\, Rx$ then $A \succ A \cup \{x\}$.

(Clearly, (GP) is in line with our interpretation of $\succsim$.)

Remark 1. If $\gtrsim$ is transitive and satisfies (GP), then $\gtrsim$ is an "extension" of R in the following sense. Let $x, y \in \Omega$, $x \neq y$ and xRy. Then, $\{x\} \succ \{y\}$. (Indeed, $\{x\} \succ \{x, y\}$ and $\{x, y\} \succ \{y\}$ by (GP).)

The second property of $\gtrsim$ which will be investigated is a *monotonocity* property.

(M) If $B, C \in 2^{\Omega}$, $B \succ C$ and $a \notin B \cup C$, then $\{a\} \cup B \gtrsim \{a\} \cup C$.

((M) says that if the decision-maker prefers that an element will be chosen from B rather than from C, then the addition of the same element a to both B and C will not reverse his preference. Thus, (M) is, also, (intuitively) acceptable. Also, (M) implies that, for every $a \in \Omega$, the function $\varphi(D) = \{a\} \cup D$ is monotonic with respect $\gtrsim$ on 2^{Ω^*}, where $\Omega^* = \Omega - \{a\}$. Hence, (M) is, indeed, a monotonicity condition.)

(GP) and (M) imply the following lemma.

LEMMA. *Assume that $\gtrsim$ is transitive. Let A be a non-empty finite subset of Ω. If $\gtrsim$ satisfies* (GP) *and* (M), *then* $A \sim \{\min(A), \max(A)\}$.

Proof. Let $A = \{a_1, ..., a_k\}$. Clearly, we may assume that $k \geqslant 3$. Also assume that $a_i R a_{i-1}$ for $i = 2, ..., k$. By repeated application of (GP) and transitivity, $\{a_k\} \succ \{a_k, ..., a_2\}$. Hence, by (M), $\{a_1, a_k\} \gtrsim A$. Similarly, $\{a_1, ..., a_{k-1}\} \succ \{a_1\}$ implies that $A \gtrsim \{a_1, a_k\}$. Thus $A \sim \{a_1, a_k\}$ and the proof is complete.

Our impossibility theorem follows now from the Lemma. We recall that a binary relation is a weak order if it is complete and transitive.

THEOREM. *If Ω contains at least six members, then there exists no weak order $\gtrsim$ on 2^{Ω} which satisfies* (GP) *and* (M).

Proof. Assume that Ω has at least six members and assume also, on the contrary, that $\gtrsim$ is a weak order on 2^{Ω} which satisfies (GP) and (M). Let $A = \{a_1, ..., a_6\}$ be a subset of Ω such that $a_i R a_{i-1}$ for $i = 2, ..., 6$. Our first claim is

$$\{a_2, a_5\} \gtrsim \{a_4\}. \tag{1}$$

Indeed, if $\{a_4\} \succ \{a_2, a_5\}$ then, by (M), $\{a_1, a_4\} \gtrsim \{a_1, a_2, a_5\}$. Hence, by the Lemma, $\{a_1, a_2, a_3, a_4\} \gtrsim \{a_1, a_2, a_3, a_4, a_5\}$, which contradicts (GP).

It follows now from (1) and Remark 1 that $\{a_2, a_5\} \succ \{a_3\}$. Hence, by (M), $\{a_2, a_5, a_6\} \gtrsim \{a_3, a_6\}$. Therefore, by the Lemma, $\{a_2, a_3\, a_4, a_5, a_6\} \gtrsim \{a_3, a_4, a_5, a_6\}$, which contradicts (GP). Thus, we have reached the desired contradiction and the proof is complete.

We recall now that a binary relation is a weak *partial* order if it is reflexive and transitive.

Remark 2. If Ω is finite then there exist weak partial orders on Ω which satisfy (GP) and (M). Indeed, for $A, B \in 2^{\Omega}$ define

$$[A \gtrsim B] \Leftrightarrow [\min(A)\, R \min(B) \text{ and } \max(A)\, R \max(B)].$$

Then, as the reader can easily verify, $\gtrsim$ is a weak partial order which satisfies (GP) and (M).

We conclude with the following remark.

Remark 3. Our assumption that R is a linear ordering of Ω does not result in any loss of generality. In fact, if R is only a weak ordering of Ω, let P be the asymmetric component of R (i.e., for $x, y \in \Omega$, $[xPy] \Leftrightarrow [xRy$ and not $yRx]$). Then, if there exist six members of Ω, $a_1,\ldots, a_6$, such that $a_i P a_{i-1}$ for $i = 2,\ldots, 6$, then our impossibility theorem remains true.

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